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Multiple Positive Solutions to a Class of Singular Boundary Value Problems for the One-Dimensional p-Laplacian

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Abstract—In this paper, we establish the existence of multiple positive solutions to the singular nonlinear boundary value problem

$$\begin{gathered}(\Phi(u'))' + q(t)f(u(t)) = 0, \qquad 0 < t < 1,\\ u(0) = 0, \qquad u(1) + B(u'(1)) = 0,\end{gathered}$$

by using the Leray-Schauder alternative and the fixed-point theorem in cones, where $\Phi(s) = |s|^{p-2}s$, $p > 1$. The singularity may appear at $u = 0$ and $t = 0$.

Keywords—Multiple positive solutions, Fixed-point theorem, p-Laplacian, Singular, Nonlinear boundary value problem.

1. INTRODUCTION

In this paper, we study the existence of multiple positive solutions to the singular boundary value problem

$$\begin{gathered}(\Phi(u'))' + q(t)f(u(t)) = 0, \qquad 0 < t < 1,\\ u(0) = 0, \qquad u(1) + B(u'(1)) = 0,\end{gathered} \tag{1.1}$$

where $\Phi(s) = |s|^{p-2}s$, $p > 1$, $q(t)$ may be singular at $t = 0$, and nonlinearity f may be singular at $u = 0$.

Recently, problem (1.1) has been studied extensively. The readers may refer to [1–14] for the details. In [1,2], the problem is not able to possess singularity. In [3–5], the problem is only able to possess singularity at $t = 0$ or $t = 1$. However, there are only a few works on the existence of multiple positive solutions to the singular boundary value problems.

In [6,7], Agarwal and O'Regan considered the singular boundary value problem

$$\begin{gathered}y''(t) + q(t)[g(y(t)) + h(y(t))] = 0, \qquad 0 < t < 1,\\ y(0) = y(1) = 0,\end{gathered}$$

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where $q(t)$ may be singular at $t = 0$ or $t = 1$, nonlinearity g may be singular at $y = 0$. They showed that this problem has single and two positive solutions by using a Leray-Schauder alternative and a fixed-point theorem in cones.

In [8–10], the authors used Krasnoselski's fixed-point theorem in cones to establish the existence of two solutions to singular boundary value problems. However, some strong integrality conditions have to be assumed on $g + h$.

In [4,5], the authors used Krasnoselski's fixed-point theorem in cones to establish the existence of multiple positive solutions when f is superlinear or sublinear. However, f is not singular at $u = 0$.

Motivated by the results mentioned above, the purpose of this paper is to establish the existence of multiple positive solutions of problem (1.1) by applying the method as used in [6,7].

Throughout this paper, we make the following hypotheses.

(H_1) B is a continuous, strictly increasing odd function defined on $(-\infty, +\infty)$.

(H_2) $q(t) : (0,1) \to (0,\infty)$ is continuous and there exists $0 \le \alpha < p - 1$ such that

$$\int_0^1 t^\alpha q(t)\,dt < \infty. \tag{1.2}$$

(H_3) $f(u) = g(u) + h(u)$ with $g > 0$ continuous and nonincreasing on $(0,\infty)$, $h \ge 0$ continuous on $[0,\infty)$, and h/g nondecreasing on $(0,\infty)$.

(H_4) There exists a constant $r > 0$ such that

$$\frac{1}{\Phi^{-1}\left(1 + (h(r))/(g(r))\right)} \int_0^r \frac{dy}{\Phi^{-1}(g(y))} > \int_0^1 \Phi^{-1}\left(\int_t^1 q(x)\,dx\right) dt, \tag{1.3}$$

where $\Phi^{-1}(u) := |u|^{1/(p-1)}\operatorname{sgn} u$ is the inverse function to $\Phi(v)$.

(H_5) Choose $a \in (0, 1/2)$ and fix it, and suppose there exists $R > r$ with

$$\frac{R}{\Phi^{-1}\left(g(R)\left[1 + h(aR)/g(aR)\right]\right)} \le \bar{M}, \tag{1.4}$$

where

$$\bar{M} = \frac{1}{2} \min_{x\in[a,1-a]} \left\{ \int_a^x \Phi^{-1}\left(\int_s^x q(t)\,dt\right) ds + \int_x^{1-a} \Phi^{-1}\left(\int_x^s q(t)\,dt\right) ds \right\}. \tag{1.5}$$

REMARK 1.1. For example, the function

$$q(t) = t^{-a}, \qquad 0 < t < 1, \quad 0 \le a < p,$$

satisfies Condition (H_2) provided $\alpha \in (a - 1, p - 1) \cap [0, p - 1)$.

REMARK 1.2. It is easy to check that Condition (H_2) implies that

$$\int_0^1 \Phi^{-1}\left(\int_t^1 q(x)\,dx\right) dt < \infty.$$

In fact,

$$\begin{aligned}\int_0^1 \Phi^{-1}\left(\int_t^1 q(x)\,dx\right) dt &\le \int_0^1 \Phi^{-1}\left(\int_t^1 \frac{x^\alpha}{t^\alpha} q(x)\,dx\right) dt \\ &\le \Phi^{-1}\left(\int_0^1 q(x)x^\alpha\,dx\right) \int_0^1 \Phi^{-1}\left(t^{-\alpha}\right) dt < \infty.\end{aligned}$$

In this paper, we say that a function $u(t)$ is a positive solution to problem (1.1) if it satisfies the following conditions:

(i) $u \in C[0,1] \cap C^1(0,1]$,

(ii) $u(t) > 0$, for all $t \in (0,1]$ and $u(0) = 0$, $u(1) + B(u'(1)) = 0$,

(iii) $\Phi(u'(t))$ is locally absolutely continuous in (0,1), and

$$\left(\Phi\left(u'\right)\right)' + q(t)f(u(t)) = 0, \qquad 0 < t < 1.$$

2. SOME PRELIMINARY RESULTS

The following lemmas will be used in our proof. The first result is a known nonlinear alternative of Leray-Schauder type [14, Theorem 1.1]. The second result is a more general fixed-point theorem in cones [7, Theorem 1.1].

LEMMA 2.1. *(See [14].) Assume Ω is a relatively open subset of a convex set K in a normal space E. Let $A : \bar{\Omega} \to K$ be a compact map with $0 \in \Omega$. Then either*

$$(\mathrm{A}_1) \qquad A \text{ has a fixed point in } \bar{\Omega}, \text{ or} \tag{2.1}$$

$$(\mathrm{A}_2) \qquad \text{there is an } x \in \partial\Omega \text{ and a } 0 < \lambda < 1 \text{ such that } x = \lambda A(x). \tag{2.2}$$

REMARK 2.1. By a map being compact, we mean it is continuous with relatively compact range.

LEMMA 2.2. *(See [7].) Let $E = (E, \|\cdot\|)$ be a Banach space and let $K \subset E$ be a cone in E, and let $\|\cdot\|$ be increasing with respect to K. Also, r, R are constants with $0 < r < R$. Suppose $A : \bar{\Omega}_R \cap K \to K$(here $\Omega_R = \{x \in E, \|x\| < R\}$) is a continuous, compact map and assume the conditions*

$$x \neq \lambda A(x), \qquad \text{for } \lambda \in [0,1) \quad \text{and} \quad x \in \partial\Omega_r \cap K \tag{2.3}$$

and

$$\|Ax\| > \|x\|, \qquad \text{for } x \in \partial\Omega_R \cap K, \tag{2.4}$$

hold. Then A has a fixed point in $K \cap \{x \in E : r \le \|x\| \le R\}$.

Let K be the cone in $C[0,1]$ given by

$$K := \{u \in C[0,1] : u(t) \text{ is a nonnegative concave function}\}.$$

The following lemma follows from the concavity of $u(t)$ on [0,1].

LEMMA 2.3. *(See Lemma 1 in [4].) Let $u \in K$ and $0 < a < 1/2$. Then*

$$\begin{array}{lll} u(t) \ge \begin{cases} \|u\|\dfrac{t}{\sigma}, & 0 \le t \le \sigma, \\ \|u\|\dfrac{1-t}{1-\sigma}, & \sigma \le t \le 1, \end{cases} & & \text{if } 0 < \sigma < 1, \\ u(t) \ge \|u\|t, & 0 \le t \le 1, & \text{if } \sigma = 1, \\ u(t) \ge \|u\|(1-t), & 0 \le t \le 1, & \text{if } \sigma = 0, \\ u(t) \ge a\|u\|, & & \text{for all } t \in [a, 1-a]. \end{array}$$

Here $\|u\| = \sup\{|u(t)| : 0 \le t \le 1\}$ and $\sigma \in [0,1]$ such that $u(\sigma) = \|u\|$.

LEMMA 2.4. *Let $h(t) : (0,1) \to (0,\infty)$ be continuous and there exists $0 \le \alpha < p-1$ such that*

$$\int_0^1 t^\alpha h(t)\, dt < \infty.$$

Then there exists a unique positive solution $V \in C[0,1] \cap C^1(0,1]$ to the problem

$$\begin{aligned} &(\Phi(V'))' + h(t) = 0, \qquad 0 < t < 1, \\ &V(0) = 0, \qquad V(1) + B(V'(1)) = 0. \end{aligned} \tag{2.5}$$

PROOF. Uniqueness. Suppose that $V_1(t)$, $V_2(t)$ are two solutions to (2.5) and let $z(t) = V_1(t) - V_2(t)$. If $z(t) \not\equiv 0$ on [0,1], since $z(0) = 0$, then $|z(\sigma)| = \max_{t\in[0,1]} |z(t)| > 0$, and $\sigma \in (0,1]$.

Without loss of generality, we assume that $z(\sigma) > 0$. Since $z(0) = 0$, then there exists an interval $(a, \sigma] \subset [0, 1]$ such that

$$z(a) = 0, \quad z(t) > 0, \qquad t \in (a, \sigma].$$

If $\sigma = 1$, then $(V_1 - V_2)'(1) \geq 0$, and so we have $B(V_1'(1)) \geq B(V_2'(1))$. It follows from (2.5) that $z(1) = -B(V_1'(1)) + B(V_2'(1)) \leq 0$, which leads to a contradiction. Then $\sigma \in (a, 1)$ and $z'(\sigma) = 0$.

Notice for $j = 1, 2$,

$$\left(\Phi\left(V_j'(x)\right)\right)' = -h(x), \qquad \text{in } (0, 1).$$

Integrating both sides of this equality over $[s, \sigma]$, $a < s \leq \sigma$,0

$$V_j'(s) = \Phi^{-1}\left(\Phi\left(V_j'(\sigma)\right) + \int_s^\sigma h(x)\,dx\right), \qquad a < s \leq \sigma.$$

Integrating both sides of the above equality from a to σ, we obtain

$$V_j(\sigma) - V_j(a) = \int_a^\sigma \Phi^{-1}\left(\Phi\left(V_j'(\sigma)\right) + \int_s^\sigma h(x)\,dx\right) ds.$$

Consequently, $z(\sigma) = V_1(\sigma) - V_2(\sigma) = 0$, which leads to a contradiction.

The proof of the uniqueness is complete.

To prove the existence of solutions, we set, for $0 < t \leq 1$,

$$y(t) := \int_0^t \Phi^{-1}\left(\int_s^t h(x)\,dx\right) ds - B \circ \Phi^{-1}\left(\int_t^1 h(x)\,dx\right) - \int_t^1 \Phi^{-1}\left(\int_t^s h(x)\,dx\right) ds.$$

Clearly, by Remark 1.2, $y(t)$ is continuous and strictly increasing in (0,1] and $y(0^+) < 0 < y(1)$. Thus, $y(t)$ has only one zero in (0,1). Let σ be the unique zero of $y(t)$ in (0,1). Then

$$\int_0^\sigma \Phi^{-1}\left(\int_s^\sigma h(x)\,dx\right) ds = B \circ \Phi^{-1}\left(\int_\sigma^1 h(x)\,dx\right) + \int_\sigma^1 \Phi^{-1}\left(\int_\sigma^s h(x)\,dx\right) ds.$$

Set

$$V(t) = \begin{cases} \displaystyle\int_0^t \Phi^{-1}\left(\int_s^\sigma h(x)\,dx\right) ds, & 0 < t \leq \sigma, \\ \displaystyle B \circ \Phi^{-1}\left(\int_\sigma^1 h(x)\,dx\right) + \int_t^1 \Phi^{-1}\left(\int_\sigma^s h(x)\,dx\right) ds, & \sigma \leq t \leq 1. \end{cases} \tag{2.6}$$

Then, V is a well-defined function on (0,1], and $V(t) > 0$ on (0,1]. Moreover,

$$V'(t) = \begin{cases} \displaystyle\Phi^{-1}\left(\int_t^\sigma h(x)\,dx\right), & 0 < t \leq \sigma, \\ \displaystyle -\Phi^{-1}\left(\int_\sigma^t h(x)\,dx\right), & \sigma \leq t \leq 1. \end{cases} \tag{2.7}$$

It follows from (H_2), for $0 < t \leq \sigma$, we have

$$\begin{aligned} 0 \leq V(t) &= \int_0^t \Phi^{-1}\left(\int_s^\sigma h(x)\,dx\right) ds \\ &\leq \int_0^t \Phi^{-1}\left(\int_s^\sigma \frac{x^\alpha}{s^\alpha} h(x)\,dx\right) ds \\ &\leq \int_0^t s^{-\alpha/(p-1)} \Phi^{-1}\left(\int_0^1 x^\alpha h(x)\,dx\right) ds \\ &= \frac{p-1}{p-1-\alpha} t^{(p-1-\alpha)/(p-1)} \Phi^{-1}\left(\int_0^1 x^\alpha h(x)\,dx\right), \end{aligned}$$

then we have $V(0) = 0$.

Similarly, we have $V(1) + B(V'(1)) = 0$. Therefore, $V(t)$ is continuous on [0,1], and

$$\begin{gathered} V(0) = 0, \qquad V(1) + B\left(V'(1)\right) = 0, \\ \left[\Phi\left(V'(t)\right)\right]' = -h(t), \qquad t \in (0,1). \end{gathered}$$

This completes the proof.

In this section, let $n \geq 4$ be a fixed natural number. We consider the modified boundary value problem

$$\begin{gathered} \left(\Phi\left(w'(t)\right)\right)' + q(t)F(u(t)) = 0, \qquad 0 < t < 1, \\ w(0) = \frac{1}{n}, \qquad w(1) + B\left(w'(1)\right) = \frac{1}{n}, \end{gathered} \tag{2.8}$$

for each $u \in K$, where $F(u) = g^*(u) + h(u)$, with

$$g^*(u) = \begin{cases} g(u), & u \geq \dfrac{1}{n}, \\ g\left(\dfrac{1}{n}\right), & u \leq \dfrac{1}{n}. \end{cases}$$

REMARK 2.2. $g^*(u) \leq g(u)$, $\forall\, u \in (0,\infty)$.

By Lemma 2.4, we have the following.

LEMMA 2.5. *For each fixed $u \in K$, the boundary value problem (2.8) has a unique solution $w \in K$ with*

$$w(t) = (\Psi u)(t),$$

where

$$(\Psi u)(t) := \begin{cases} \dfrac{1}{n} + \displaystyle\int_0^t \Phi^{-1}\left(\int_s^{\sigma_u} q(x)F(u(x))\,dx\right) ds, & 0 \leq t \leq \sigma_u, \\ \dfrac{1}{n} + B\circ\Phi^{-1}\left(\displaystyle\int_{\sigma_u}^1 q(x)F(u(x))\,dx\right) + \displaystyle\int_t^1 \Phi^{-1}\left(\int_{\sigma_u}^s q(x)F(u(x))\,dx\right) ds, & \\ \qquad\qquad \sigma_u \leq t \leq 1, & \end{cases} \tag{2.9}$$

for $u \in K$, where $\sigma_u \in (0,1)$ is the unique solution of the equation

$$\begin{aligned} z_0(\tau) &:= \frac{1}{n} + \int_0^\tau \Phi^{-1}\left(\int_s^\tau q(x)F(u(x))\,dx\right) ds \\ &= \frac{1}{n} + B\circ\Phi^{-1}\left(\int_\tau^1 q(x)F(u(x))\,dx\right) \\ &\quad + \int_t^1 \Phi^{-1}\left(\int_\tau^s q(x)F(u(x))\,dx\right) ds := z_1(\tau), \end{aligned} \tag{2.10}$$

for $0 \leq \tau \leq 1$.

From the definition of w and Ψ for $u \in K$, we have

(i)

$$w'(t) = \begin{cases} \Phi^{-1}\left(\displaystyle\int_t^{\sigma_u} q(x)F(u(x))\,dx\right) \geq 0, & 0 < t \leq \sigma_u, \\ -\Phi^{-1}\left(\displaystyle\int_{\sigma_u}^t q(x)F(u(x))\,dx\right) \leq 0, & \sigma_u \leq t \leq 1, \end{cases}$$

(ii) $(\Phi(w'(t)))' = -q(t)F(u(t))$ *in (0,1) and* $w(0) = 1/n$, $w(1) + B(w'(1)) = 1/n$,

(iii) $w = \Psi u \in K$, $\|w\| = w(\sigma_u)$.

This shows that $w(t)$ is a solution to (2.8) and a concave function defined on [0,1].

LEMMA 2.6. *Let $w_i(t)$ be a solution to problem (2.8) with $F = F_i$, $i = 1, 2$. If $F_1 \le F_2$, then $w_1(t) \le w_2(t)$.*

PROOF. Let $z(t) = w_1(t) - w_2(t)$. If the lemma is not true, similar to the proof of Lemma 2.4, there exists an interval $(a, \sigma] \subset (0, 1)$ such that $z(t) > 0$ in $(a, \sigma]$, and

$$z(a) = 0, \qquad z(\sigma) = \max_{t \in [0,1]} z(t) > 0, \qquad z'(\sigma) = 0.$$

Notice that

$$(\Phi(w_1'(t)))' = -q(t)F_1(u(t)) \ge -q(t)F_2(u(t)) = (\Phi(w_2'(t)))', \qquad t \in (a, \sigma].$$

Integrating both sides of this equality over $[s, \sigma], a < s \le \sigma$, we get

$$-\Phi(w_1'(s)) + \Phi(w_1'(\sigma)) \ge -\Phi(w_2'(s)) + \Phi(w_2'(\sigma)), \qquad a < s \le \sigma,$$

i.e.,

$$z'(s) = w_1'(s) - w_2'(s) \le 0, \qquad a < s \le \sigma.$$

Consequently, $z(\sigma) \le z(a) = 0$, which leads to a contradiction. The lemma is proved.

Let $V_M(t)$ be a positive solution to problem (2.5) with $h(t) = Mq(t)$ $(M > 0)$ and $V_m(t)$ be a positive solution to problem (2.5) with $h(t) = mq(t)$ $(m > 0)$.

By Lemmas 2.4–2.6, we have the following remarks.

REMARK 2.3. Let $w(t)$ be a solution to problem (2.8) with $F(u) \le M$. Then $w(t) \le 1/n + V_M(t)$, i.e., $(\Psi u)(t) \le 1/n + V_M(t)$.

REMARK 2.4. Let $w(t)$ be a solution to problem (2.8) with $F(u) \ge m$. Then $w(t) \ge 1/n + V_m(t)$, i.e., $(\Psi u)(t) \ge 1/n + V_m(t)$.

LEMMA 2.7. *Let $[a, 1] \subset (0, 1]$ be a compact interval, and let $w(t)$ be a solution to problem (2.8) with $F(u) \le M$, then*

$$|w'(t)| \le C(a, M), \qquad a \le t \le 1,$$

where M is a positive constant, $C(a, M)$ is a positive constant dependent of a, M.

PROOF. We can obtain that

$$w'(t) = \Phi^{-1}\left(\tau + \int_t^1 q(s)F(u(s))\, ds\right), \qquad a \le t \le 1, \tag{2.11}$$

where $\tau = \Phi(w'(1))$ is a solution of the equation

$$\int_a^1 \Phi^{-1}\left(\tau + \int_r^1 q(s)F(u(s))\, ds\right) dr = w(1) - w(a). \tag{2.12}$$

By the first mean value theorem, there exists a $\xi \in [a, 1]$ such that

$$\Phi^{-1}\left(\tau + \int_\xi^1 q(s)F(u(s))\, ds\right) = \frac{w(1) - w(a)}{1 - a},$$

i.e.,

$$\tau = -\int_\xi^1 q(s)F(u(s))\, ds + \Phi\left(\frac{w(1) - w(a)}{1 - a}\right).$$

By Remark 2.3, $w(t) \le V_M(t) + 1/n \le V_M(t) + 1$, then there is a $C > 0$ such that

$$|\tau| \le \int_a^1 q(s)M\, ds + \Phi\left(\frac{w(1) + w(a)}{1 - a}\right) \le C \tag{2.13}$$

and

$$\left|\tau + \int_t^1 q(s)F(u(s))\, ds\right| \le C. \tag{2.14}$$

From (2.11), (2.13), and (2.14), we have

$$|w'(t)| \le C(a, M), \qquad a \le t \le 1,$$

where $C(a, M)$ is a positive constant dependent of a, M.

LEMMA 2.8. *For any bounded and closed $\Omega \subset K$, the set $\Psi(\Omega)$ is equicontinuous on [0,1].*

PROOF. Let $M > 0$ such that $F(u) \le M$ for $u \in \Omega$. For any $\varepsilon > 0$, from the continuity of $V_M(t)$ on [0,1] and $V_M(0) = 0$, then there is a $\delta_1 \in (0, 1/4)$ such that

$$V_M(t) < \frac{\varepsilon}{2}, \qquad \text{for } t \in [0, 2\delta_1].$$

Let $u \in \Omega$. Since $(\Psi u)(t) - 1/n \le V_M(t)$, then for any $t_1, t_2 \in [0, 2\delta_1]$, $|t_1 - t_2| < \delta_1$,

$$\begin{aligned} |(\Psi u)(t_1) - (\Psi u)(t_2)| &= \left|(\Psi u)(t_1) - \frac{1}{n} + \frac{1}{n} - (\Psi u)(t_2)\right| \\ &\le \left|(\Psi u)(t_1) - \frac{1}{n}\right| + \left|(\Psi u)(t_2) - \frac{1}{n}\right| \\ &\le V_M(t_1) + V_M(t_2) < \varepsilon. \end{aligned}$$

By Lemma 2.7, $|(\Psi u)'(t)| \le C(\delta_1, M) =: L$, for $t \in [\delta_1, 1]$.

Put $\delta_2 = \varepsilon / L$. Then for $t_1, t_2 \in [\delta_1, 1]$ with $|t_1 - t_2| < \delta_2$, we have

$$|(\Psi u)(t_1) - (\Psi u)(t_2)| \le L|t_1 - t_2| < \varepsilon.$$

Set $\delta = \min\{\delta_1, \delta_2\}$. Then for $t_1, t_2 \in [0, 1]$ with $|t_1 - t_2| < \delta$, we obtain

$$|(\Psi u)(t_1) - (\Psi u)(t_2)| < \varepsilon.$$

This shows that $\Psi(\Omega)$ is equicontinuous on [0,1].

LEMMA 2.9. *For any bounded and closed $\Omega \subset K$, the mapping $\Psi : \Omega \to K$ is continuous.*

PROOF. Let $M > 0$ such that $F(u) \le M$ for $u \in \Omega$. Assume that $u_0, u_j \in \Omega$ and $\|u_j - u_0\| \to 0$ as $j \to \infty$. Then we have

$$(\Psi u_j)(t) = \begin{cases} \dfrac{1}{n} + \displaystyle\int_0^t \Phi^{-1}\left(\int_s^{\sigma_{u_j}} q(x)F(u_j(x))\,dx\right) ds, & 0 \le t \le \sigma_{u_j}, \\ \dfrac{1}{n} + B \circ \Phi^{-1}\left(\displaystyle\int_{\sigma_{u_j}}^1 q(x)F(u_j(x))\,dx\right) & \\ \quad + \displaystyle\int_t^1 \Phi^{-1}\left(\int_{\sigma_{u_j}}^s q(x)F(u_j(x))\,dx\right) ds, & \sigma_{u_j} \le t \le 1, \end{cases}$$

where σ_{u_j}, $j = 0, 1, \dots$, satisfies the following equation:

$$\begin{aligned} &\int_0^{\sigma_{u_j}} \Phi^{-1}\left(\int_s^{\sigma_{u_j}} q(x)F(u_j(x))\,dx\right) ds \\ &= B \circ \Phi^{-1}\left(\int_{\sigma_{u_j}}^1 q(x)F(u_j(x))\,dx\right) + \int_{\sigma_{u_j}}^1 \Phi^{-1}\left(\int_{\sigma_{u_j}}^s q(x)F(u_j(x))\,dx\right) ds. \end{aligned} \tag{2.15^j}$$

Suppose that $\sigma^* \in [0, 1]$ is an accumulation point of $\{\sigma_{u_j}\}$. Then there is a subsequence of $\{\sigma_{u_j}\}$,$\{\sigma_{u_{j(m)}}\}$, which converges to σ^*. Inserting $u_{j(m)}$ and $\sigma_{u_{j(m)}}$ into $(2.15)^{j(m)}$ and then $m \to \infty$, we obtain

$$\begin{aligned} \int_0^{\sigma^*} \Phi^{-1}\left(\int_s^{\sigma^*} q(x)F(u_0(x))\,dx\right) ds &= B \circ \Phi^{-1}\left(\int_{\sigma^*}^1 q(x)F(u_0(x))\,dx\right) \\ &\quad + \int_{\sigma^*}^1 \Phi^{-1}\left(\int_{\sigma^*}^s q(x)F(u_0(x))\,dx\right) ds. \end{aligned}$$

This shows that $\sigma^* = \sigma_{u_0}$, and hence, $\sigma_{u_j} \to \sigma_{u_0}$. Thus, we use the dominated convergence theorem to obtain

$$\lim_{j\to\infty} (\Psi u_j)(t) = \lim_{j\to\infty} \begin{cases} \dfrac{1}{n} + \displaystyle\int_0^t \Phi^{-1}\left(\int_s^{\sigma_{u_j}} q(x)F(u_j(x))\,dx\right) ds, & 0 \le t \le \sigma_{u_j}, \\ \dfrac{1}{n} + B\circ\Phi^{-1}\left(\displaystyle\int_{\sigma_{u_j}}^1 q(x)F(u_j(x))\,dx\right) & \\ \qquad + \displaystyle\int_t^1 \Phi^{-1}\left(\int_{\sigma_{u_j}}^s q(x)F(u_j(x))\,dx\right) ds, & \sigma_{u_j} \le t \le 1 \end{cases}$$
$$= (\Psi u_0)(t), \qquad t \in [0,1].$$

This shows that Ψ is continuous from Ω to K.

Combining Lemmas 2.5–2.9, we have the following.

LEMMA 2.10. *$\Psi : K \to K$ is completely continuous.*

3. MAIN RESULTS

In this section, we examine the singular nonlinear boundary value problem

$$\begin{gathered} (\Phi(u'))' + q(t)f(u(t)) = 0, \qquad 0 < t < 1, \\ u(0) = 0, \qquad u(1) + B(u'(1)) = 0. \end{gathered} \tag{3.1}$$

THEOREM 3.1. *Suppose Conditions (H_1)–(H_4) are satisfied, then (3.1) has a solution $u \in C[0,1] \cap C^1(0,1]$ with $u > 0$ on $(0,1]$ and $\|u\| < r$.*

PROOF. To show the existence of the solution described in the statement of Theorem 3.1, we apply Lemma 2.1 first. We can choose $\varepsilon > 0$, and $\varepsilon < r$ such that

$$\frac{1}{\Phi^{-1}(1 + h(r)/g(r))} \int_\varepsilon^r \frac{dy}{\Phi^{-1}(g(y))} > \int_0^1 \Phi^{-1}\left(\int_t^1 q(x)\,dx\right) dt. \tag{3.2}$$

Let $n_0 \in \{1, 2, \dots\}$ be chosen so that $1/n_0 < \varepsilon$. Let $N^+ = \{n_0, n_0+1, \dots\}$. We first show that the following boundary value problem:

$$\begin{gathered} (\Phi(u'))' + q(t)f(u(t)) = 0, \qquad 0 < t < 1, \\ u(0) = \frac{1}{n}, \quad u(1) + B(u'(1)) = \frac{1}{n}, \qquad n \in N^+ \end{gathered} \tag{3.3^n}$$

has a solution $u_n(t)$, $n \in N^+$, $u_n(t) > 1/n$ on $(0,1]$, and $|u_n(t)| < r$.

To show $(3.3)^n$ has such a solution, $\forall\, n \in N^+$, we will deal with the modified boundary value problem

$$\begin{gathered} (\Phi(u'))' + q(t)F(u(t)) = 0, \qquad 0 < t < 1, \\ u(0) = \frac{1}{n}, \quad u(1) + B(u'(1)) = \frac{1}{n}, \qquad n \in N^+, \end{gathered} \tag{3.4^n}$$

where F is defined by (2.8).

Fix $n \in N^+$. Let $\Psi : \bar{\Omega}_r \to K$ be defined by

$$(\Psi u)(t) = \begin{cases} \dfrac{1}{n} + \displaystyle\int_0^t \Phi^{-1}\left(\int_s^{\sigma_u} q(x)F(u(x))\,dx\right) ds, & 0 \le t \le \sigma_u, \\ \dfrac{1}{n} + B\circ\Phi^{-1}\left(\displaystyle\int_{\sigma_u}^1 q(x)F(u(x))\,dx\right) & \\ \qquad + \displaystyle\int_t^1 \Phi^{-1}\left(\int_{\sigma_u}^s q(x)F(u(x))\,dx\right) ds, & \sigma_u \le t \le 1, \end{cases} \tag{3.5}$$

where $\sigma_u \in (0,1)$ is the unique solution of the equation

$$\begin{aligned} z_0(\tau) &:= \frac{1}{n} + \int_0^\tau \Phi^{-1}\left(\int_s^\tau q(x)F(u(x))\,dx\right) ds \\ &= \frac{1}{n} + B\circ\Phi^{-1}\left(\int_\tau^1 q(x)F(u(x))\,dx\right) + \int_t^1 \Phi^{-1}\left(\int_\tau^s q(x)F(u(x))\,dx\right) ds \\ &:= z_1(\tau), \qquad 0\le\tau\le 1. \end{aligned}$$

Similar to the proof of Lemmas 2.5–2.9, we can obtain that $\Psi : \bar{\Omega}_r \to K$ is completely continuous.

We first show

$$u \neq \lambda\Psi u, \qquad \text{for } \lambda\in(0,1), \quad u\in\partial\Omega_r. \tag{3.6}$$

Suppose this is false, suppose that there exist a $\lambda\in(0,1)$ and $u\in\partial\Omega_r$ with $u=\lambda\Psi u$. Then we have

$$\begin{aligned} &-(\Phi(u'))' = \lambda^{p-1}q(t)F(u(t)), \qquad 0<t<1, \\ &u(0)=\frac{\lambda}{n}, \quad u(1)+\lambda B\left(\frac{u'(1)}{\lambda}\right)=\frac{\lambda}{n}, \qquad n\in N^+. \end{aligned} \tag*{$(3.7)^n$}$$

Clearly there exists $\sigma_n\in(0,1)$ with $u'(t)\ge 0$ on $(0,\sigma_n)$, $u'(t)\le 0$ on $(\sigma_n,1)$, and $u(\sigma_n)=\|u\|=r$.

Also notice that

$$F(u(t))\le g(u(t))+h(u(t)), \qquad t\in(0,1);$$

then for $z\in(0,1)$, we have

$$-(\Phi(u'(z)))' \le g(u(z))\left\{1+\frac{h(u(z))}{g(u(z))}\right\}q(z). \tag{3.8}$$

Integrate from $t\,(0<t\le\sigma_n)$ to σ_n to obtain

$$u'(t)\le\Phi^{-1}\left(\left\{1+\frac{h(r)}{g(r)}\right\}\int_t^{\sigma_n} g(u(z))q(z)\,dz\right), \tag{3.9}$$

so we obtain

$$\frac{u'(t)}{\Phi^{-1}(g(u(t)))}\le\Phi^{-1}\left(\left\{1+\frac{h(r)}{g(r)}\right\}\right)\Phi^{-1}\left(\int_t^{\sigma_n} q(z)\,dz\right), \tag{3.10}$$

and then integrate from 0 to σ_n to obtain

$$\int_{\lambda/n}^r \frac{dy}{\Phi^{-1}(g(y))}\le\Phi^{-1}\left(\left\{1+\frac{h(r)}{g(r)}\right\}\right)\int_0^{\sigma_n}\Phi^{-1}\left(\int_t^{\sigma_n} q(z)\,dz\right)dt. \tag{3.11}$$

Consequently,

$$\int_\varepsilon^r \frac{dy}{\Phi^{-1}(g(y))}\le\Phi^{-1}\left(\left\{1+\frac{h(r)}{g(r)}\right\}\right)\int_0^{\sigma_n}\Phi^{-1}\left(\int_t^{\sigma_n} q(z)\,dz\right)dt, \tag{3.12}$$

and so,

$$\int_\varepsilon^r \frac{dy}{\Phi^{-1}(g(y))}\le\Phi^{-1}\left(\left\{1+\frac{h(r)}{g(r)}\right\}\right)\int_0^1\Phi^{-1}\left(\int_t^1 q(z)\,dz\right)dt. \tag{3.13}$$

This contradicts (3.2) and consequently (3.6) is true.

Now Lemma 2.1 implies Ψ has a fixed point $u_n(t)\in\bar{\Omega}_r$, i.e., $1/n\le\|u_n\|\le r$ (note if $\|u_n\|=r$, then following essentially the same argument from (3.8)–(3.13) will yield a contradiction). Since $u_n\ge 1/n$, we can obtain that $u_n(t)$ is a solution of $(3.3)^n$ also.

From (H_3), for $r > 0$, $g(u_n(t)) \geq g(r)$, $f(u_n) = h(u_n) + g(u_n) \geq g(r)$. Then by Remark 2.4, we have

$$u_n(t) \geq \frac{1}{n} + V_{g(r)}(t), \qquad 0 \leq t \leq 1. \tag{3.14}$$

REMARK 3.1. Notice that $V_{g(r)}(t) > 0$ on $(0,1]$, then $u_n(t) > 0$, $t \in (0,1]$.

Next we will show that

$$\{u_n\}_{n \in N^+} \text{ is a bounded, equicontinuous family on } [0,1]. \tag{3.15}$$

Returning to (3.8) (with u replaced by u_n), we have

$$-\left(\Phi(u_n'(z))\right)' \leq g\left(u_n(z)\right)\left\{1 + \frac{h\left(u_n(z)\right)}{g\left(u_n(z)\right)}\right\} q(z). \tag{3.16}$$

Since $u_n(t) \geq 1/n$ on $[0,1]$, there exists $\sigma_n \in (0,1)$ with $u_n'(t) \geq 0$ on $(0,\sigma_n)$, $u_n'(t) \leq 0$ on $(\sigma_n, 1)$ and $u_n(\sigma_n) = \|u_n\| \leq r$.

Integrating (3.16) from $t\,(0 < t < \sigma_n)$ to σ_n, we have

$$\frac{u_n'(t)}{\Phi^{-1}\left(g\left(u_n(t)\right)\right)} \leq \Phi^{-1}\left(\left\{1 + \frac{h(r)}{g(r)}\right\}\right) \Phi^{-1}\left(\int_t^{\sigma_n} q(z)\,dz\right). \tag{3.17}$$

We now claim that there exists $a_0 > 0$, with

$$a_0 < \inf\left\{\sigma_n : n \in N^+\right\} \leq 1.$$

If this is not true, then there exists a subsequence S of N^+ with $\sigma_n \to 0$ as $n \to \infty$ in S. Now integrate (3.17) from 0 to σ_n to obtain

$$\int_0^{u_n(\sigma_n)} \frac{dy}{\Phi^{-1}(g(y))} \leq \Phi^{-1}\left(1 + \frac{h(r)}{g(r)}\right) \int_0^{\sigma_n} \Phi^{-1}\left(\int_t^{\sigma_n} q(x)\,dx\right) dt + \int_0^{1/n} \frac{dy}{\Phi^{-1}(g(y))}, \tag{3.18}$$

for $n \in S$. Since $\sigma_n \to 0$ as $n \to \infty$ in S, we have from (3.18) that $u_n(\sigma_n) \to 0$ as $n \to \infty$ in S. However, since the maximum of u_n on [0,1] occurs at σ_n, we have $u_n \to 0$ in $C[0,1]$ as $n \to \infty$ in S. This contradicts (3.14).

This implies

$$\frac{u_n'(t)}{\Phi^{-1}\left(g\left(u_n(t)\right)\right)} \leq \Phi^{-1}\left(\left\{1 + \frac{h(r)}{g(r)}\right\}\right) \Phi^{-1}(W(t)), \qquad \text{for } t \in (0,\sigma_n), \tag{3.19}$$

where $W(t) = \int_{\min(t,a_0)}^1 q(z)\,dz$. By Remark 1.2, $\Phi^{-1}(W) \in L^1[0,1]$.

When $\sigma_n \leq t \leq 1$, we have $-u_n'(t) \leq -u_n'(1) = B^{-1}(u_n(1) - 1/n) \leq B^{-1}(r)$. So we have

$$\frac{-u_n'(t)}{\Phi^{-1}\left(g\left(u_n(t)\right)\right)} \leq \frac{B^{-1}(r)}{\Phi^{-1}(g(r))}, \qquad \text{for } t \in (\sigma_n, 1). \tag{3.20}$$

Now, (3.19) and (3.20) imply

$$\frac{|u_n'(t)|}{\Phi^{-1}\left(g\left(u_n(t)\right)\right)} \leq \Phi^{-1}\left(\left\{1 + \frac{h(r)}{g(r)}\right\}\right) \left[\Phi^{-1}(W(t)) + \Phi(g(r) + h(r))B^{-1}(r)\right], \quad \text{for } t \in (0,1). \tag{3.21}$$

Let $I : [0,\infty) \to [0,\infty)$ be defined by

$$I(z) = \int_0^z \frac{dy}{\Phi^{-1}(g(y))}.$$

Note, $I : [0,\infty) \to [0,\infty)$ is an increasing map and $I(\infty) = \infty$, since $g(u) > 0$ is nonincreasing on $(0,\infty)$ with I continuous on $[0, B]$ for any $B > 0$. Notice

$$\{I(u_n)\}_{n\in N^+} \text{ is a bounded, equicontinuous family on } [0,1].$$

The equicontinuity follows from (here $t, s \in [0,1]$)

$$\begin{aligned} |I(u_n(t)) - I(u_n(s))| &= \left| \int_s^t \frac{u_n'(z)}{\Phi^{-1}(g(u_n(z))} \, dz \right| \\ &\le \Phi^{-1}\left(1 + \frac{h(r)}{g(r)}\right) \left| \int_s^t \left[\Phi^{-1}(W(z)) + \Phi(g(r) + h(r))B^{-1}(r)\right] \, dz \right|. \end{aligned}$$

This inequality, the uniform continuity of I^{-1}, and

$$|u_n(t) - u_n(s)| = \left|I^{-1}(I(u_n(t))) - (I(u_n(s)))\right|,$$

now establish (3.15).

The Arzela-Ascoli theorem guarantees the existence of a subsequence $N \subset N^+$, $u \in C[0,1]$ with u_n converging uniformly on $[0,1]$ to u as $n \to \infty$, for $n \in N$. Then by (3.14), $u(t) \ge V_{g(r)}(t)$, on $[0,1]$. In particular $u(t) > 0$ on $(0,1]$. Fixing $t \in (0,1]$, we have

$$u_n(1) = u_n(t) + \int_t^1 \Phi^{-1}\left[\Phi\left(u_n'(1)\right) + \int_s^1 q(x) f\left(u_n(x)\right) \, dx\right] ds. \tag{3.22}$$

Since $|u_n'(1)| = -u_n'(1) = B^{-1}(u_n(1) - 1/n) \le B^{-1}(r)$. Thus, $\{u_n'(1)\}_{n\in N}$ has a convergent subsequence; for convenience let $\{u_n'(1)\}_{n\in N}$ denote this subsequence also, and let $r_0 \in \mathbf{R}$ be its limit. Now for the above fixed $t \in (0,1]$, let $n \to \infty$ through N (we note here that qf is uniformly continuous on compact subsets of $[t,1] \times (0,r]$) to obtain

$$u(1) = u(t) + \int_t^1 \Phi^{-1}\left[\Phi\left(r_0\right) + \int_s^1 q(x) f(u(x)) \, dx\right] ds. \tag{3.23}$$

We can do this argument for each $t \in (0,1]$ and so

$$u'(t) = \Phi^{-1}\left[\Phi(r_0) + \int_t^1 q(x) f(u(x)) \, dx\right], \qquad 0 < t \le 1,$$

thus, $r_0 = u'(1)$. Then we have $(\Phi(u'))' + q(t)f(u(t)) = 0$ for $0 < t < 1$, $u(0) = u(1) + B(u'(1)) = 0$. Finally, it is easy to see that $\|u\| < r$ (note, if $\|u\| = r$, then following essentially the same argument from (3.8)–(3.13) will yield a contradiction).

Thus, we have proved that problem (1.1) has one positive solution $u(t) \in C[0,1] \cap C^1(0,1]$, and $\|u\| < r$.

THEOREM 3.2. *Let (H_1)–(H_5) hold. Then problem (3.1) has a solution $u \in C[0,1] \cap C^1(0,1]$ with $u > 0$ on $(0,1]$ and $r < \|u\| \le R$.*

PROOF. To show the existence of the solution described in the statement of Theorem 3.2, we will apply Lemma 2.2.

From (1.3), there exists $\varepsilon > 0\,(\varepsilon < r)$, which satisfies (3.2). Let $n_0 \in \{1, 2, \dots\}$ such that $1/n_0 < \varepsilon$, $1/n_0 < aR$. Let $N^+ = \{n_0, n_0 + 1, \dots\}$.

First, we will show that

$$\begin{aligned} &(\Phi\left(u'\right))' + q(t) f(u(t)) = 0, \qquad 0 < t < 1, \\ &u(0) = \frac{1}{n}, \quad u(1) + B\left(u'(1)\right) = \frac{1}{n}, \qquad n \in N^+, \end{aligned} \tag{3.3^n}$$

has a solution $u_n(t)$, $n \in N^+$, $u_n(t) > 1/n$ on $(0,1)$, and $r < \|u_n\| < R$.

To show $(3.3)^n$ has such a solution, $\forall n \in N^+$, we will deal with the modified boundary value problem

$$\begin{gathered}(\Phi(u'))' + q(t)F(u(t)) = 0, \qquad 0 < t < 1,\\ u(0) = \frac{1}{n}, \quad u(1) + B(u'(1)) = \frac{1}{n}, \qquad n \in N^+.\end{gathered} \tag{3.4^n}$$

Fix $n \in N^+$. Let $\Psi : K \to K$ be defined by

$$(\Psi u)(t) = \begin{cases} \dfrac{1}{n} + \displaystyle\int_0^t \Phi^{-1}\left(\int_s^{\sigma_u} q(x)F(u(x))\,dx\right) ds, & 0 \le t \le \sigma_u,\\ \dfrac{1}{n} + B \circ \Phi^{-1}\left(\displaystyle\int_{\sigma_u}^1 q(x)F(u(x))\,dx\right) & \\ \qquad + \displaystyle\int_t^1 \Phi^{-1}\left(\int_{\sigma_u}^s q(x)F(u(x))\,dx\right) ds, & \sigma_u \le t \le 1,\end{cases} \tag{3.24}$$

where σ_u is the unique solution of the equation

$$\begin{aligned} z_0(\tau) &:= \frac{1}{n} + \int_0^\tau \Phi^{-1}\left(\int_s^\tau q(x)F(u(x))\,dx\right) ds\\ &= \frac{1}{n} + B \circ \Phi^{-1}\left(\int_\tau^1 q(x)F(u(x))\,dx\right) + \int_\tau^1 \Phi^{-1}\left(\int_\tau^s q(x)F(u(x))\,dx\right) ds\\ &:= z_1(\tau), \qquad 0 \le \tau \le 1.\end{aligned}$$

From Lemma 2.10, we obtain that $\Psi : K \to K$ is completely continuous.

We first show

$$u \ne \lambda \Psi u, \qquad \text{for } \lambda \in (0,1), \quad u \in \partial\Omega_r \cap K, \tag{3.25}$$

where Ω_r is defined above. Similar to the proof of (3.7)–(3.13), we can show that (3.25) is true.

Next we will show

$$\|\Psi u\| > \|u\|, \qquad \forall u \in \partial\Omega_R \cap K. \tag{3.26}$$

To see this, let $u \in \partial\Omega_R \cap K$ such that $\|u\| = R$.

Since $u \in K$, then by Lemma 2.3, $u(s) \ge aR$ for $s \in [a, 1-a]$, so we have

$$g^*(u(s)) + h(u(s)) = g(u(s)) + h(u(s)).$$

Note in particular that $u(s) \in [aR, R]$, for $s \in [a, 1-a]$, with σ_u as defined in (3.5), we obtain

$$\begin{aligned} 2(\Psi u)(\sigma_u) &\ge \int_0^{\sigma_u} \Phi^{-1}\left(\int_s^{\sigma_u} q(x)f(u(x))\,dx\right) ds + \int_{\sigma_u}^1 \Phi^{-1}\left(\int_{\sigma_u}^s q(x)f(u(x))\,dx\right) ds\\ &> \int_a^{\sigma_u} \Phi^{-1}\left(\int_s^{\sigma_u} g(u(x))\left\{1 + \frac{h(u(x))}{g(u(x))}\right\} q(x)\,dx\right) ds\\ &\quad + \int_{\sigma_u}^{1-a} \Phi^{-1}\left(\int_{\sigma_u}^s g(u(x))\left\{1 + \frac{h(u(x))}{g(u(x))}\right\} q(x)\,dx\right) ds\\ &\ge \int_a^{\sigma_u} \Phi^{-1}\left(\int_s^{\sigma_u} g(R)\left\{1 + \frac{h(aR)}{g(aR)}\right\} q(x)\,dx\right) ds\\ &\quad + \int_{\sigma_u}^{1-a} \Phi^{-1}\left(\int_{\sigma_u}^s g(R)\left\{1 + \frac{h(aR)}{g(aR)}\right\} q(x)\,dx\right) ds\\ &= \Phi^{-1}\left(g(R)\left\{1 + \frac{h(aR)}{g(aR)}\right\}\right)\\ &\quad \times \left[\int_a^{\sigma_u} \Phi^{-1}\left(\int_s^{\sigma_u} q(x)\,dx\right) ds + \int_{\sigma_u}^{1-a} \Phi^{-1}\left(\int_{\sigma_u}^s q(x)\,dx\right) ds\right]\\ &\ge 2\bar{M}\Phi^{-1}\left(g(R)\left\{1 + \frac{h(aR)}{g(aR)}\right\}\right)\\ &\ge 2R, \qquad \text{when } \sigma_u \in [a, 1-a],\end{aligned}$$

$$
\begin{aligned}
(\Psi u)(\sigma_u) &> \int_a^{1-a} \Phi^{-1}\left(\int_s^{1-a} q(x) f(u(x))\,dx\right) ds \\
&= \int_a^{1-a} \Phi^{-1}\left(\int_s^{1-a} g(u(x))\left\{1+\frac{h(u(x))}{g(u(x))}\right\} q(x)\,dx\right) ds \\
&\geq \int_a^{1-a} \Phi^{-1}\left(g(R)\left\{1+\frac{h(aR)}{g(aR)}\right\}\int_s^{1-a} q(x)\,dx\right) ds \\
&= \Phi^{-1}\left(g(R)\left\{1+\frac{h(aR)}{g(aR)}\right\}\right)\int_a^{1-a} \Phi^{-1}\left(\int_s^{1-a} q(x)\,dx\right) ds \\
&\geq \bar{M}\Phi^{-1}\left(g(R)\left\{1+\frac{h(aR)}{g(aR)}\right\}\right) \\
&\geq R, \qquad \text{when } \sigma_u > 1-a, \\
(\Psi u)(\sigma_u) &> \int_a^{1-a} \Phi^{-1}\left(\int_a^{s} q(x) f(u(x))\,dx\right) ds \\
&= \int_a^{1-a} \Phi^{-1}\left(\int_a^{s} g(u(x))\left\{1+\frac{h(u(x))}{g(u(x))}\right\} q(x)\,dx\right) ds \\
&\geq \int_a^{1-a} \Phi^{-1}\left(g(R)\left\{1+\frac{h(aR)}{g(aR)}\right\}\int_a^{s} q(x)\,dx\right) ds \\
&= \Phi^{-1}\left(g(R)\left\{1+\frac{h(aR)}{g(aR)}\right\}\right)\int_a^{1-a} \Phi^{-1}\left(\int_a^{s} q(x)\,dx\right) ds \\
&\geq \bar{M}\Phi^{-1}\left(g(R)\left\{1+\frac{h(aR)}{g(aR)}\right\}\right) \\
&\geq R, \qquad \text{when } \sigma_u < a.
\end{aligned}
$$

This shows that

$$\|\Psi u\| > R = \|u\|, \qquad \text{for } u \in \partial\Omega_R \cap K. \tag{3.27}$$

Now Lemma 2.2 implies Ψ has a fixed point $u_n(t) \in K \cap (\bar{\Omega}_R \setminus \Omega_r)$, $r < \|u_n\| \leq R$. Clearly, $\|u_n\| \neq r$. Consequently, $(3.4)^n$ has a solution $u_n(t) \in C[0,1] \cap C^1(0,1]$, $u_n(t) \in K$.

Moreover, similar to the proof to get (3.14), we obtain

$$u_n(t) \geq \frac{1}{n} + V_{g(R)}(t), \qquad 0 \leq t \leq 1, \tag{3.28}$$

which shows that $(3.3)^n$ has a positive solution $u_n(t)$.

By the same way as above, $u_n(t)$ have subsequences N of N^+, with $u_n(t)$ converging uniformly on $[0,1]$ to $u(t)$ as $n \to \infty$ through N. It is easy to show that $u(t) \in C[0,1] \cap C^1(0,1]$ is a positive solution of (3.1) and $r < \|u\| \leq R$.

Thus, the proof of Theorem 3.2 is complete.

THEOREM 3.3. *Assume (H_1)–(H_5) hold. Then (3.1) has two solutions $u_1, u_2 \in C[0,1] \cap C^1(0,1]$ with $u_1 > 0, u_2 > 0$ on $(0,1]$ and $\|u_1\| < r < \|u_2\| \leq R$.*

PROOF. The existence of u_1 follows from Theorem 3.1, and the existence of u_2 follows from Theorem 3.2.

EXAMPLE 3.1. The singular boundary value problem

$$
\begin{gathered}
(\Phi(u'))' + \sigma\left(u^{-\alpha} + u^{\beta} + 1\right) = 0, \qquad 0 < t < 1, \\
u(0) = 0, \qquad u(1) + B(u'(1)) = 0
\end{gathered} \tag{3.29}
$$

has two solutions $u_1, u_2 \in C[0,1] \cap C^1(0,1]$ with $u_1 > 0$, $u_2 > 0$ on (0,1] and $\|u_1\| < 1 < \|u_2\|$. Here

$$\alpha > 0, \qquad \beta > p-1, \qquad 0 < \sigma < \frac{1}{3}\left(\frac{p}{p-1+\alpha}\right)^{p-1}.$$

To see this, we will apply Theorem 3.3 with

$$q(s) = \sigma, \qquad g(u) = u^{-\alpha}, \qquad h(u) = u^{\beta} + 1.$$

Clearly, (H_1)–(H_3) hold. Also note

$$\int_0^1 \Phi^{-1}\left(\int_t^1 \sigma\, dx\right) dt = \int_0^1 [\sigma(1-t)]^{1/(p-1)}\, dt = \sigma^{1/(p-1)} \frac{p-1}{p}.$$

Consequently, (1.3) holds (with $r = 1$), since

$$\frac{1}{\Phi^{-1}\left(1 + h(r)/g(r)\right)} \int_0^r \frac{dy}{\Phi^{-1}(g(y))} = \left(\frac{1}{3}\right)^{1/(p-1)} \int_0^1 y^{\alpha/(p-1)}\, dy = \left(\frac{1}{3}\right)^{1/(p-1)} \frac{p-1}{p-1+\alpha},$$

then

$$\left(\frac{1}{3}\right)^{1/(p-1)} \frac{p-1}{p-1+\alpha} > \sigma^{1/(p-1)} \frac{p-1}{p}.$$

Finally, note that (since $\beta > p-1$, set $a = 1/4$),

$$\begin{aligned}
&\lim_{R\to\infty} \frac{R}{\Phi^{-1}(R^{-\alpha}[1 + ((1/4)R)^{\alpha+\beta} + ((1/4)R)^{\alpha}])} \\
&= \lim_{R\to\infty} \frac{R}{(R^{-\alpha} + (1/4)^{\alpha+\beta}R^{\beta} + (1/4)^{\alpha})^{1/(p-1)}} \\
&= 0,
\end{aligned}$$

so there exists $R > 1$ with (1.4) holding. The result now follows from Theorem 3.3.

EXAMPLE 3.2. Consider the singular boundary value problem

$$\begin{gathered}
\left(\Phi\left(u'\right)\right)' + \sigma t^{-m}\left(u^{-\alpha} + u^{\beta}\right) = 0, \qquad 0 < t < 1, \\
u(0) = 0, \qquad u(1) + B\left(u'(1)\right) = 0,
\end{gathered} \tag{3.30}$$

with $0 \le m < p$, $\sigma > 0$, $\alpha > 0$, $\beta > p-1$.

Set

$$g(u) = u^{-\alpha}, \qquad h(u) = u^{\beta}, \qquad q_1(t) = t^{-m}, \qquad q(t) = \sigma q_1(t),$$

$$b_1 := \int_0^1 \Phi^{-1}\left(\int_t^1 q_1(x)\, dx\right) dt.$$

Then $b_0 = \sigma^{1/(p-1)} b_1$.

Applying Theorem 3.3, in the same way as in Example 3.1, we can find that (3.30) has two positive solutions if there exists $r > 0$ such that

$$\sigma < \left[\frac{p-1}{b_1(\alpha+p-1)}\right]^{p-1} \frac{r^{\alpha+p-1}}{1+r^{\alpha+\beta}}. \tag{3.31}$$

Set

$$T(x) := \frac{x^{\alpha+p-1}}{1+x^{\alpha+\beta}}, \qquad x > 0,$$

then

$$T(x_0) = \sup_{x\in(0,\infty)} T(x), \qquad x_0 = \left(\frac{\alpha+p-1}{\beta+1-p}\right)^{1/(\alpha+\beta)}.$$

We choose $r = x_0$ such that (3.31) holds. Obviously, (H_1)–(H_5) in Theorem 3.3 are satisfied. Thus, (3.30) has two solutions $u_1, u_2 \in C[0,1] \cap C^1(0,1]$ with $u_1 > 0$, $u_2 > 0$ on (0,1] and $\|u_1\| < r = x_0 < \|u_2\|$.

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